Extremal and Probabilistic Graph Theory
March 1st, Tuesday

- **Notations.**

  Graph $G = (V, E)$, where $E \subset V \times V$;

  Hypergraph $H = (V, E)$, where $E \subset \bigcup_{i \geq 2} \underbrace{V \times \ldots \times V}_{i}$;

  δ minimum degree;

  Δ maximum degree;

  $d = \frac{\sum_v d_v}{n}$ average degree.

- **Definition 1.** A k-graph $H = (V, E)$ is a k-uniform hypergraph where each edge $e \in E$ has k vertices.

- **Remark.** We also view e as a subset. $|e| = \#$ vertices in e.

- **Definition 2.** For a subset $S \in V(H)$, define the degree of S, $d_H((S) = \#$ edges with $S \subset e$.

- **Prop 1.** For any hypergraph H and $r \geq 2$,

$$\sum_{e \in E(H)} \binom{|e|}{r} = \sum_{S \in \binom{V}{r}} d_H(S)$$

  where $\binom{V}{r} = \{$ all subset of size r in V $\}$.

  Proof. By double counting $\#$ of (S,e) where $|S| = r$ and $S \subset e$. ∎

- **Definition 3.** $K_n^{(k)}$ denotes the complete k-graph on n vertices;

  $K^{(k)}(V_1, V_2, \ldots, V_k)$ denotes the complete k-partite k-graph on parts $V_1, V_2, \ldots, V_k$;

  $K_{k:t}$ denotes a copy of $K^{(k)}(V_1, V_2, \ldots, V_k)$, where each $|V_i| = t$.

- **Definition 4.** Let H be a hypergraph, a link hypergraph of a set $S \subset V(H)$ in H is a hypergraph with vertex set $V \setminus S$, and edge set $\{e \setminus S \colon S \subset e \text{ in } H\}$.

- **Definition 5.** Let $\mathscr{F}$ be a family of k-graphs. A k-graph H is $\mathscr{F}$-free if No $F \in \mathscr{F}$ is contained in H. When $\mathscr{F} = \{G\}$, call it G-free.

- **Definition 6.** Let $ex_k(n, \mathscr{F}) =$ max $e(H)$ over all n-vertex $\mathscr{F}$-free k-graph H. $ex_k(n, \mathscr{F})$ is called Turán function of $\mathscr{F}$. We write $ex(n, \mathscr{F})$ when $k = 2$. Turán density of $\mathscr{F}$

$$\pi(\mathscr{F}) = \lim_{n \to \infty} \frac{ex_k(n, \mathscr{F})}{\binom{n}{k}}$$

  .

- **Prop 2.** $\pi(\mathscr{F})$ exists for all $\mathscr{F}$.

  Proof. Let$\pi_n(\mathscr{F}) = \frac{ex_k(n,\mathscr{F})}{\binom{n}{k}}$. Consider any n-vertex $\mathscr{F}$-free k-graph H. Let us count # of (e,T), where $T \subset V(H)$ is of size $n-1$ and $e \subset T$. Fixing e, there are $\binom{n-k}{n-k-1}$ choices of T and we have

$$\#(e,T) = \sum_{e\in E} \binom{n-k}{n-k-1}$$

  . On the other hand, there are at most $ex_k(n-1,\mathscr{F})$ edges in T for each T, since T is $\mathscr{F}$-free. Hence we have $\#(e,T) \leq n\ ex_k(n-1,\mathscr{F})$. We have the following inequality

$$\frac{ex_k(n,\mathscr{F})}{\binom{n}{k}} \leq \frac{n}{(n-k)(\binom{n}{k})} ex_k(n-1,\mathscr{F}) = \frac{ex_k(n-1,\mathscr{F})}{\binom{n-1}{k}}$$

  By the definition of $\pi_n(\mathscr{F})$, we have $\pi_n\mathscr{F} \leq \pi_{n-1}(\mathscr{F})$. So, $\pi_n(\mathscr{F})$ is a non-increasing sequence, and $\pi_n(\mathscr{F}) \geq 0$. Thus

$$\pi(\mathscr{F}) = \lim_{n\to\infty} \pi_n(\mathscr{F})$$

  exists. ∎

- **Remark.** $0 \leq \pi(\mathscr{F}) \leq 1$.

- **Supersaturation Lemma.** Fix $k \geq 2$ and F be a k-graph. For $\forall\epsilon > 0$, $\exists\delta > 0$, such that if H is an n-vertex k-graph with at least $ex_k(n,F) + \epsilon n^k$ edges, then H contains at least $\delta\binom{n}{\nu(F)}$ copies of F, where $\nu(F) = \#$ vertices of F.

- **Prop 3.** $\forall\ t \geq k \geq 2$, $\pi(K_t^{(k)}) \leq 1 - \frac{1}{\binom{t}{k}}$.

  Proof: $\forall\ n \geq t$, $\pi_n(K_t^{(k)}) \leq \pi_t(K_t^{(k)}) = 1 - \frac{1}{\binom{t}{k}}$. ∎

- **Prop 4.** $\frac{5}{9} \leq \pi(K_4^{(3)}) \leq \frac{1}{\sqrt{2}}$.

  Proof. 1) upper bound. To prove $\pi(K_4^{(3)}) \leq \frac{1}{\sqrt{2}}$, we need to show all n-vertex $K_4(3)$-free 3-graph H has no more than $\frac{1}{\sqrt{2}}\binom{n}{3} + o(n^3)$ edges.

  Fact. There are at most 3 edges on any 4 vertices .

  We will count #subgraph L isomorphic to {123,124} on 4 vertices. By the fact above,each four ser has at most 3 edges and hence has at most 3 copies of L, so we have $\#L \leq 3\binom{n}{4}$. On the other hand, $\#L = \sum_{S\in\binom{V}{2}} \binom{d_H(S)}{2}$. So we have $3\binom{n}{4} \geq \#L = \sum_{S\in\binom{V}{2}} \binom{d_H(S)}{2} \geq \binom{n}{2}\binom{\frac{\sum_S d_H(S)}{\binom{n}{2}}}{2} = \binom{n}{2}\binom{\frac{3e(H)}{\binom{n}{2}}}{2} \Rightarrow e(H) \leq \frac{1}{\sqrt{2}}\binom{n}{3} + o(n^3)$. (the second inequality holds by Cauchy-Schwarz)

  2) lower bound. We need find a n-vertex $K_4^{(3)}$-free 3-graph H with $\frac{5}{9}\binom{n}{3} + o(n^3)$ edges. $V(H) = X \bigcup Y \bigcup Z$, where $X \bigcap Y = Y \bigcap Z = Z \bigcap X = \emptyset$ and $|X| = |Y| = |Z| = \frac{|V(H)|}{3} = \frac{n}{3}$. $E(H)$={all edges of type xyz, x_1x_2y, y_1y_2z, z_1z_2x, $x \in X$, $y \in Y$, $z \in Z$ }. Check that H is $K_4^{(3)}$-free and H has $(\frac{n}{3})^3 + 3\binom{\frac{n}{3}}{2}(\frac{n}{3}) = \frac{5}{9}\binom{n}{3}$ edges. ∎

- **Conjecture.** $\pi(K_4^{(3)}) = \frac{5}{9}$.